



Problem 3. «The BSW Conjecture»

Problem for a special prize!

Notation. Let $\text{GF}(q)$ be the finite field of order q , and $\text{GF}(q)^*$ be the set of its nonzero elements. The quadratic character χ of $\text{GF}(q)$ takes the value 1 for nonzero squares, the value -1 for nonzero nonsquares, and $\chi(0) = 0$.

Conjecture 1. Let $q > 31$ be an odd prime power. Take a map ψ of $\text{GF}(q)^*$ that is an involution, pairs every square with a non-square, and satisfies

$$\chi((x - y)(x - \psi y)(\psi x - y)(\psi x - \psi y)) = -1$$

whenever $y \notin \{x, \psi x\}$. Then $\psi(x) = n/x$ for some non-square n .

The case $q \equiv 1 \pmod{4}$. Blokhuis, Seress and Wilbrink showed¹ the Conjecture for every prime power $q \equiv 1 \pmod{4}$ with $q > 31$. The case $q \equiv 3 \pmod{4}$ is open.

Geometric formulation. Work in $\text{PG}(2, q)$ with q odd, and use the conic

$$C: \quad xy = z^2.$$

Remember, that

- A *tangent* is a line meeting C in exactly one point.
- A *passant* (external line) is a line containing no point of C .
- An *exterior point* is a point off C that lies on a tangent.
- An *interior point* is a point off C that lies on no tangent.

Conjecture 2. Let $q \equiv 3 \pmod{4}$ and $q > 31$. Let S be a set of $(q + 1)/2$ exterior points of C such that every line through two distinct points of S is a passant. Then S is exactly the set of exterior points on some passant.

Note that Conjectures 1 and 2 are equivalent.

The problem. Please, prove the Conjecture 1 for an arbitrary q (general problem). Prove it for an infinite class of prime power q 's or for finitely many values of q .

¹See A. Blokhuis, Á. Seress, and H. A. Wilbrink, Characterization of complete exterior sets of conics, *Combinatorica* **12** (1992), no. 2, 143–147. DOI 10.1007/BF01204717.