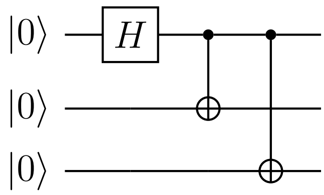
Problem 6. «Quantum entanglement»

The Nobel Prize in Physics in 2022 was awarded to researchers who experimentally investigated quantum *entanglement*. One of their studies was devoted to a Greenberger–Horne–Zeilinger state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, which is an entangled state of three qubits. This state can be created using the following quantum circuit:



After the measurement, the probability to find the system described by $|GHZ\rangle$ in the state $|000\rangle$ or in the state $|111\rangle$ is equal to $1/2$.

When we make measurements in quantum physics, we are able to make *post-selection*. For example, if we post-select the events when the first qubit was in state $|0\rangle$, the second and the third qubits will also be found in the state $|0\rangle$ for sure, this is actually what entanglement means. We also see that the post-selection destroys entanglement of two remaining qubits.

Q1 But what will happen, if we post-select the events when the 1rst qubit is in the Hadamard state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$? How can we perform this kind of post-selection if the result of each measurement of a qubit state can be only 0 or 1 and we can only post-select these events? Will the two remaining qubits be entangled after post-selection? Design the circuit which will provide an answer.

Q2 **Problem for a special prize!** There are two different classes of three-qubit entanglement. One of them is

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$

and the other is

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

Discuss the possible ideas how the difference between these states can be found with the usage of post-selection and measurement. Don't forget that you need to verify entanglement for both types of states!

Turn to the next page.

Remark. Let us briefly formulate the key points of quantum circuits. A qubit is a two-level quantum mechanical system whose state $|\psi\rangle$ is the superposition of basis quantum states $|0\rangle$ and $|1\rangle$. The superposition is written as $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$, where α_0 and α_1 are complex numbers, called amplitudes, that possess $|\alpha_0|^2 + |\alpha_1|^2 = 1$. The amplitudes α_0 and α_1 have the following physical meaning: after the measurement of a qubit which has the state $|\psi\rangle$, it will be found in the state $|0\rangle$ with probability $|\alpha_0|^2$ and in the state $|1\rangle$ with probability $|\alpha_1|^2$. Note that we can measure qubit, initially given in the state $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$, in other basis, for example Hadamard basis $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. In order to do this, we consider the state in the form $|\psi\rangle = \alpha_0' |+\rangle + \alpha_1' |-\rangle$, where complex amplitudes α_0', α_1' have the same physical meaning as α_0 and α_1. Then we can calculate the probability that the qubit will be in the state $|+\rangle$ or $|-\rangle$ after the measurement and consider the process of post-selection in this case. In order to operate with multi-qubit systems, we consider the bilinear operation $\otimes : |x\rangle, |y\rangle \to |x\rangle \otimes |y\rangle$ on $x, y \in \{0, 1\}$ which is defined on pairs $|x\rangle, |y\rangle$, and by bilinearity is expanded on the space of all linear combinations of $|0\rangle$ and $|1\rangle$. When we have two qubits in states $|\psi\rangle$ and $|\varphi\rangle$ correspondingly, the state of the whole system of these two qubits is $|\psi\rangle \otimes |\varphi\rangle$. In general, for two qubits we have $|\psi\rangle = \alpha_{00}|0\rangle \otimes |0\rangle + \alpha_{01} |0\rangle \otimes |1\rangle + \alpha_{10} |1\rangle \otimes |0\rangle + \alpha_{11} |1\rangle \otimes |1\rangle$. The physical meaning of complex numbers α_{ij} is the same as for one qubit, so we have the essential restriction $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$. We use more brief notation $|a\rangle \otimes |b\rangle \equiv |ab\rangle$. By induction, this process is expanded on the case of three qubits and more. Mathematically, the entanglement of n-qubits state means that we can not consider this state in the form $|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$, where $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are some states of m and $n - m$ qubits, correspondingly. In order to verify your circuits, you can use different quantum circuit simulators, for example `https://algassert.com/quirk`.