Problem 8. «Quantum error correction»

The procedure of error correction is required for quantum computing due to intrinsic errors in quantum gates. One of approaches to quantum error correction is to encode quantum information in three-qubit states, i. e. $\alpha_0 |0\rangle + \alpha_1 |1\rangle \to \alpha_0 |000\rangle + \alpha_1 |111\rangle$. Below are **Problems for a special prize!**

Q1 Design a circuit which implements such encoding.

Q2 Design a circuit which restores the initial state of the three-qubit system, if a single bit-flip error $|0\rangle \leftrightarrow |1\rangle$ occurs in one of three qubits. Hint: use two additional qubits and three-qubit Toffoli gates.

Q3 What will happen, if the quantum gates used for error correction are imperfect? What will be the threshold for gate fidelity, when the error correction will stop working?

Remark. A qubit is a two-level quantum mechanical system whose state $|\psi\rangle$ is the superposition of basis quantum states $|0\rangle$ and $|1\rangle$. The superposition is written as $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$, where α_0 and α_1 are complex numbers that possess $|\alpha_0|^2 + |\alpha_1|^2 = 1$. The amplitudes α_0 and α_1 have the following physical meaning: after the measurement of a qubit which has the state $|\psi\rangle$, it will be found in the state $|0\rangle$ with probability $|\alpha_0|^2$ and in the state $|1\rangle$ with probability $|\alpha_1|^2$. In order to operate with multi-qubit systems, we consider the bilinear operation $\otimes : |x\rangle, |y\rangle \to |x\rangle \otimes |y\rangle$ on $x, y \in \{0,1\}$ which is defined on pairs $|x\rangle, |y\rangle$, and by bilinearity is expanded on the space of all linear combinations of $|0\rangle$ and $|1\rangle$. When we have two qubits in states $|\psi\rangle$ and $|\varphi\rangle$ correspondingly, the state of the whole system of these two qubits is $|\psi\rangle \otimes |\varphi\rangle$. In general, for two qubits we have $|\psi\rangle = \alpha_{00}|0\rangle \otimes |0\rangle + \alpha_{01} |0\rangle \otimes |1\rangle + \alpha_{10} |1\rangle \otimes |0\rangle + \alpha_{11} |1\rangle \otimes |1\rangle$. The physical meaning of complex numbers α_{ij} is the same as for one qubit, so we have the essential restriction $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$. We use more brief notation $|a\rangle \otimes |b\rangle \equiv |ab\rangle$. For the case of multi-qubit systems with n qubits the general form of the state is $|\psi\rangle = \sum\limits_{(i_1 i_2 \ldots i_n) \in \{0,1\}^n} \alpha_{i_1 i_2 \ldots i_n} |i_1 i_2 \ldots i_n\rangle$.

Pauli-X gate	$|x\rangle$ — $\boxed{X}$ — $|x \oplus 1\rangle$	acts on a single qubit in the state $|x\rangle$, $x \in \{0,1\}$
Pauli-Z gate	$|x\rangle$ — $\boxed{Z}$ — $(-1)^x |x\rangle$	acts on a single qubit in the state $|x\rangle$, $x \in \{0,1\}$
Hadamard gate	$|x\rangle$ — $\boxed{H}$ — $\frac{|0\rangle + (-1)^x |1\rangle}{\sqrt{2}}$	acts on a single qubit in the state $|x\rangle$, $x \in \{0,1\}$
controlled NOT (CNOT) gate	$|x\rangle \to |x\rangle$; $|y\rangle \to |y \oplus x\rangle$	acts on a pair of qubits in the states $|x\rangle, |y\rangle$, $x, y \in \{0,1\}$
SWAP gate	$|x\rangle \to |y\rangle$; $|y\rangle \to |x\rangle$	acts on a pair of qubits in the states $|x\rangle, |y\rangle$, $x, y \in \{0,1\}$
Toffoli gate	$|x\rangle \to |x\rangle$; $|y\rangle \to |y\rangle$; $|z\rangle \to |z \oplus (x \cdot y)\rangle$	acts on a triple of qubits in the states $|x\rangle, |y\rangle, |z\rangle$, $x, y, z \in \{0,1\}$