Problem 10. «A fixed element»

A polynomial $f(X_1, \ldots, X_n) \in \mathbb{F}_2[X_1, \ldots, X_n]$ is called *reduced* if the degree of each X_i in f is at most 1. For $0 \leqslant r \leqslant n$, the rth order Reed — Muller code of length 2^n, denoted by $R(r,n)$, is the $\mathbb{F}_2$-space of all reduced polynomials in $X_1, \ldots, X_n$ of total degree $\leqslant r$. We also define $R(-1,n) = \{0\}$.

The general linear group $\mathrm{GL}(n, \mathbb{F}_2)$ acts on $R(r,n)$ naturally: Given $A \in \mathrm{GL}(n, \mathbb{F}_2)$ and $f(X_1, \ldots, X_n) \in R(r,n)$, Af is defined to be the reduced polynomial obtained from $f((X_1, \ldots, X_n)A)$ by replacing each power X_i^k ($k \geqslant 2$) with X_i. Consequently, $\mathrm{GL}(n, \mathbb{F}_2)$ acts on the quotient space $R(r,n)/R(r-1,n)$.

Let $A \in \mathrm{GL}(n, \mathbb{F}_2)$ be such that its characteristic polynomial is a primitive irreducible polynomial over $\mathbb{F}_2$. Prove that the only element in $R(r,n)/R(r-1,n)$, where $0 < r < n$, fixed by the action of A is 0.