Problem 8. «High-nonlinear functions»

One of interesting classes of one-to-one vectorial Boolean functions of the form $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$, where n is even, is the set of functions such that $F^{-1} = F$. Does this class contain a function with nonlinearity not less than $2^{n-1} - 2^{n/2}$?

Remark. Recall several definitions.

- A vectorial Boolean function $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ can be represented as the set of its n coordinate Boolean functions: $F = (f_1, f_2, \ldots, f_n)$, where $f_1, \ldots, f_n : \mathbb{F}_2^n \to \mathbb{F}_2$;
- the Hamming distance $dist(f, g)$ between two Boolean function $f, g : \mathbb{F}_2^n \to \mathbb{F}_2$ is equal to the number of vectors $x \in \mathbb{F}_2^n$ such that $f(x) \neq g(x)$.
- Nonlinearity nl_F of F is equal to

$$\min_{b \in \mathbb{F}_2^n, b \neq 0} \min_{a \in \mathbb{F}_2^n, c \in \mathbb{F}_2} dist(b \cdot F, \ell_{a,c})$$

where $b \cdot F = b_1 f_1 \oplus b_2 f_2 \oplus \ldots \oplus b_n f_n$ and $\ell_{a,c}(x) = a_1 x_1 \oplus a_2 x_2 \oplus \ldots \oplus a_n x_n \oplus c$.