Task 10. «Linear subspaces»

For constructing a new secret sharing scheme Mary has to solve the following task on binary vectors. Let n be an integer number, $n \geqslant 2$. Let $\mathbb{F}_2^{2n}$ be a $2n$-dimensional vector space over $\mathbb{F}_2$, where $\mathbb{F}_2 = \{0, 1\}$ is a prime field of characteristic 2.

Do there exist subsets $L_1, \ldots, L_{2^n+1}$ of $\mathbb{F}_2^{2n}$ such that the following conditions hold

1. L_i is a linear subspace of dimension n for every $i \in \{1, \ldots, 2^n + 1\}$;
2. $L_i \cap L_j = \{\mathbf{0}\}$ for all $i, j \in \{1, \ldots, 2^n + 1\}$, $i \neq j$;
3. $L_1 \cup \ldots \cup L_{2^n+1} = \mathbb{F}_2^{2n}$?

If «yes», show how to construct these subspaces for an arbitrary integer n.

Remark I. Recall several definitions and notions. Each element $x \in \mathbb{F}_2^k$ is a binary vector of length k, i. e. $x = (x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in \mathbb{F}_2$. For two vectors x and y of length k their sum is $x \oplus y = (x_1 \oplus y_1, \ldots, x_k \oplus y_k)$, where $\oplus$ stands for XOR operation. Let $\mathbf{0}$ be the zero element of the vector space, i. e. vector with all-zero coordinates. A nonempty subset $L \subseteq \mathbb{F}_2^k$ is called a *linear subspace* if for any $x, y \in L$ it holds $x \oplus y \in L$. It is easy to see that zero vector belongs to every linear subspace. A linear subspace L of $\mathbb{F}_2^k$ has *dimension* n if it contains exactly 2^n elements.

Remark II. For example the case $n = 2$ we consider together. In the vector space $\mathbb{F}_2^4$ we can choose the following 5 required subspaces:

$$\begin{aligned}
L_1 &= \{(0000), (0001), (1110), (1111)\};\\
L_2 &= \{(0000), (0010), (1001), (1011)\};\\
L_3 &= \{(0000), (0011), (0100), (0111)\};\\
L_4 &= \{(0000), (0101), (1000), (1101)\};\\
L_5 &= \{(0000), (0110), (1010), (1100)\}.
\end{aligned}$$