Problem 9. «Bases»

Problem for a special prize! Let us consider the vector space $\mathbb{F}_2^r$ consisting of all binary vectors of length r. For any d vectors $x^i = (x_1^i, \ldots, x_r^i)$, $i = 1, \ldots, d$, $d > 0$, it is defined the componentwise product of these vectors equal to $(x_1^1 \ldots x_1^d, \ldots, x_r^1 \ldots x_r^d)$. The empty product (when no element is involved in it) equals the all-ones vector.

Let $s \geqslant d > 1$ be positive integers and let r be defined by the formula $r = \sum_{i=0}^{d} \binom{s}{i}$, where $\binom{s}{i}$ denotes the binomial coefficient. Let $\mathcal{B}$ be a basis of the vector space $\mathbb{F}_2^r$, and let $\mathcal{F} \subseteq \mathbb{F}_2^r$ be a family of s binary vectors such that all possible componentwise products of up to d vectors from the family $\mathcal{F}$ (including the empty product) form the basis $\mathcal{B}$.

Given s, d, r defined above, describe all (or at least some) bases $\mathcal{B}$ for which such family $\mathcal{F}$ exists or prove that such bases do not exist.

Suggest practical applications of such bases.

Example. Let $s = 2$, $d = 2$ and $r = 4$. Consider the following family of 2 vectors $\mathcal{F} = \{(1100), (0110)\}$. Then all componentwise products of 0, 1 and 2 vectors from the family $\mathcal{F}$ form the basis $\mathcal{B} = \{(1111), (1100), (0110), (0100)\}$ of the vector space $\mathbb{F}_2^4$.